Classification of Convex Ancient Solutions to Free Boundary Curve Shortening Flow in Convex Domains

Nathan Burns

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ANU Geometric Analysis Seminar

Mean Curvature Flow and Ancient Solutions

Let M^n be a compact n-dimensional manifold.

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Definition

A one-parameter family of immersions $X : M^n \times [0, T) \to \mathbb{R}^{n+1}$ is said to evolve with *mean curvature flow* if

$$\partial_t X = \vec{H} \tag{1}$$

where $\vec{H}$ is the mean curvature vector on $\mathcal{M}_t := X(M^n, t)$.

Existence of Solutions

Theorem (Short Time Existence)

Let $X_0 : M^n \to \mathbb{R}^{n+1}$ be a smooth immersion. Then there exists ϵ and a smooth solution $X : M^n \times [0, \epsilon) \to \mathbb{R}^{n+1}$ to the mean curvature flow with $X(\cdot, 0) = X_0$.

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Theorem (Long Time Existence)

Let $X : M^n \times [0, T) \to \mathbb{R}$ be a solution to the mean curvature flow on a maximal time interval, where $T < \infty$. Then

$$\sup_{M^n \times [0,T)} |A|^2 = \infty,$$

where A is the second fundamental form of $\mathcal{M}_t$.

Ancient Solutions

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Ancient solutions to MCF model singularities:

- Type I singularities rescale to ancient solutions.
- Type II singularities rescale to eternal solutions (solutions that exist for all time).

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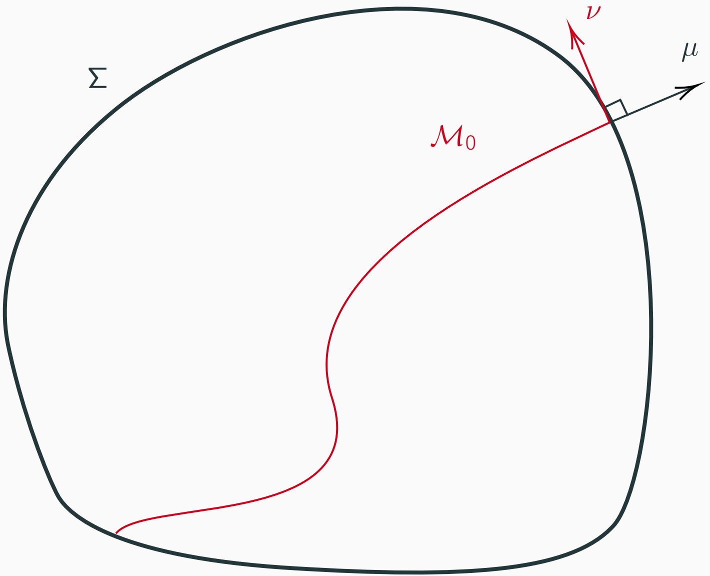
Definition

A one-parameter family of immersions $X : M^n \times [0,T) \to \mathbb{R}^{n+1}$ satisfies the **free boundary mean curvature flow** if

$$\partial_t X = \vec{H}, \quad x \in \mathring{M}^n$$
$$X(x,t) \subseteq \Sigma, \quad x \in \partial M^n$$
$$\langle \nu, \mu \circ X \rangle (x,t) = 0, \quad x \in \partial M^n$$

where ν is the normal field to $\mathcal{M}_t$, and μ is the normal field to Σ.

Free Boundary Mean Curvature Flow



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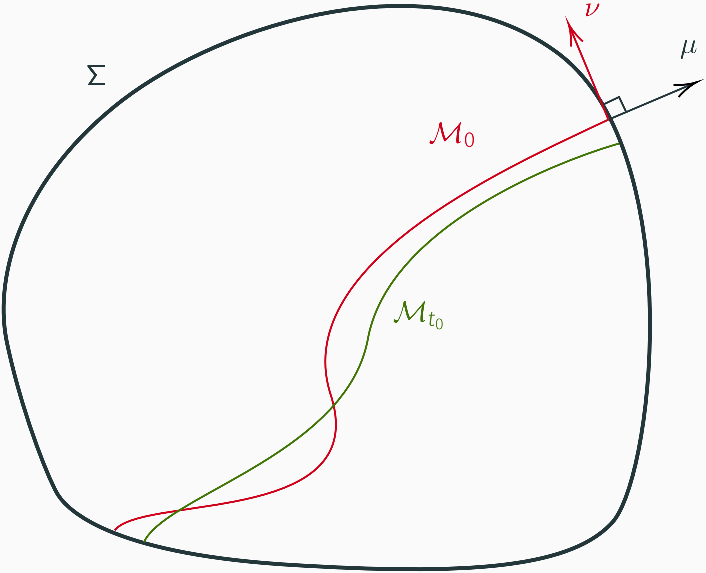

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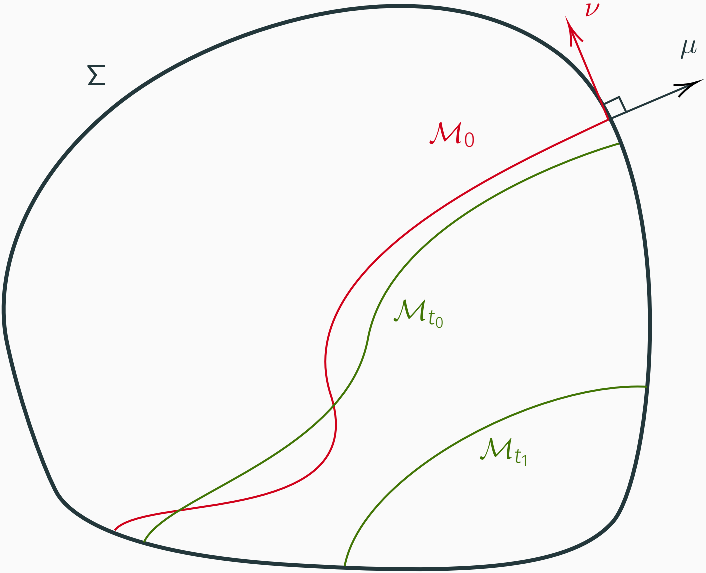

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- (Langford-Zhu '23) If $\Omega \subset \mathbb{R}^2$ is a C^2 convex domain, and $\mathcal{M}_0$ is a properly embedded curve which meets $\partial\Omega$ orthogonally. Then either,

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  1. $\mathcal{M}_t$ converges smoothly to chord in Ω and $T = \infty$.
  2. $\mathcal{M}_t$ converges to a point on $\partial\Omega$ with rescaled image the unit semi-circle.

Classifying Ancient Solutions

Known Classifications

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Ancient Solutions to the Curve Shortening Flow:

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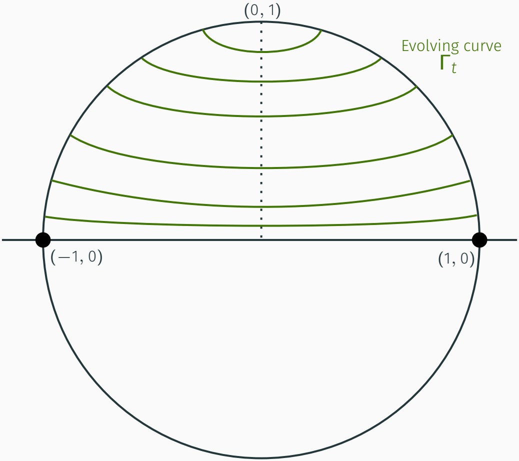
- (Bourni-Langford ’22) There is a unique, non-trivial, convex-embedded ancient solution to the free boundary curve shortening flow in S^1.

Ancient Solution in the Circle



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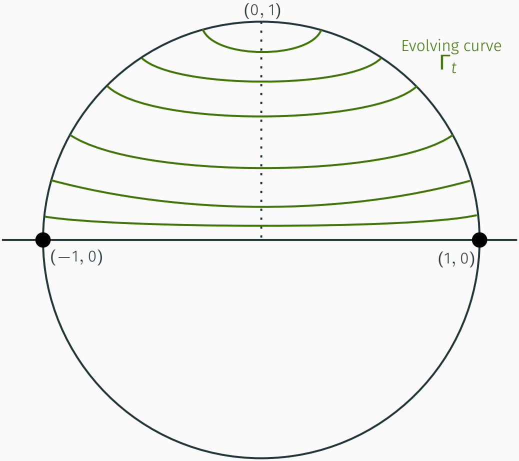

Goal: Extend this result to *any* compact, strictly convex free boundary in $\mathbb{R}^2$.

Main Result

Theorem (B.-Bourni-Catron ’24)

Let Ω be a bounded, connected, strictly convex domain in $\mathbb{R}^2$. Modulo time-translation, for each diameter of Ω (that is, a line segment intersecting the boundary of Ω orthogonally), there exists precisely two convex, locally uniformly convex, ancient solutions to the free boundary curve shortening flow in Ω, one lying on each side of the diameter.

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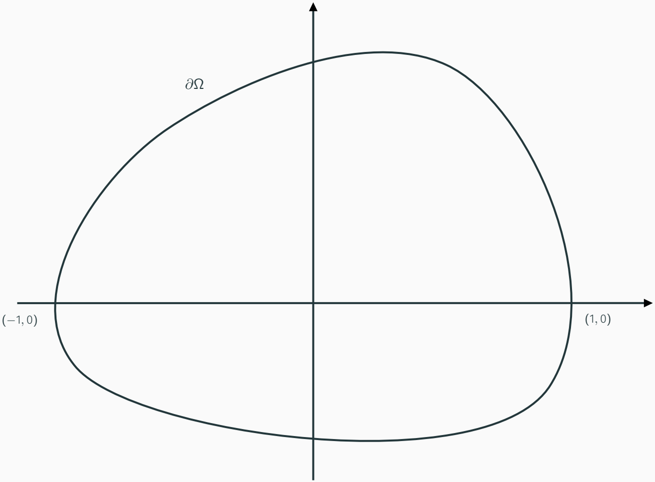
Proof Idea:

- Existence: Take a sequence of old-but-not-ancient solutions $\{\Gamma_t\}_{t\in[\alpha_\rho,0)}$.
- Uniqueness: Look at the asymptotic behavior of the quantity $e^{-\lambda_0^2 t}y(x,t)$, and use the strong maximum principle.

Existence of an Ancient Solution

Old-But-Not-Ancient Solutions



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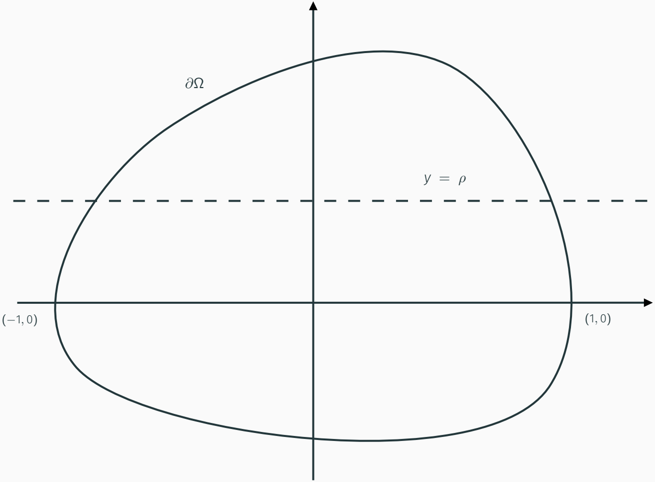

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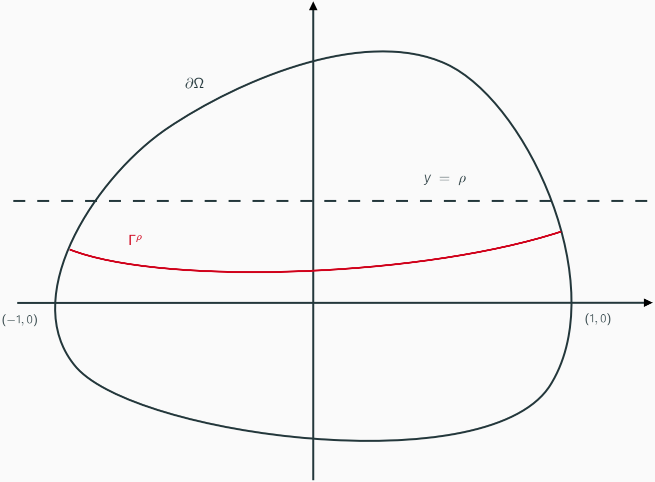

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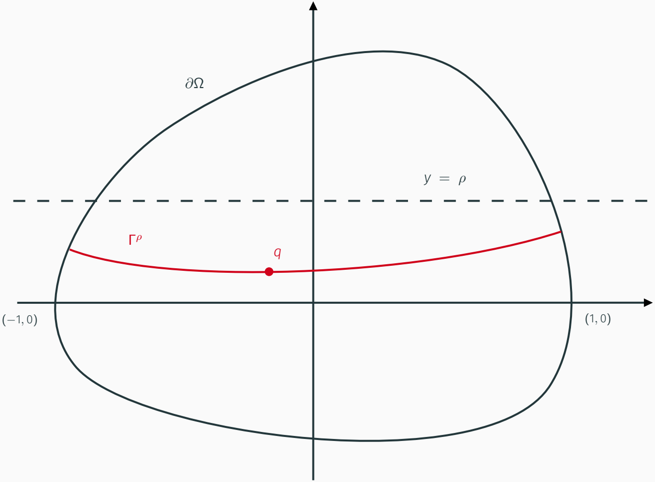

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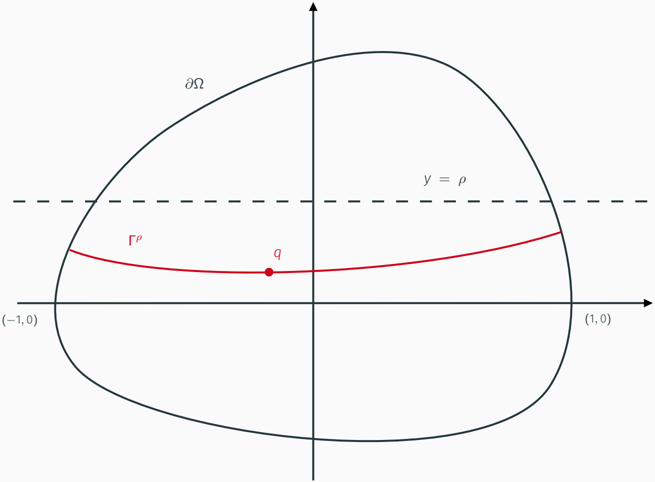

q is the unique point for which the curvature is minimised, i.e., $\nabla\kappa = 0$.

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To do this, we use *barriers*.

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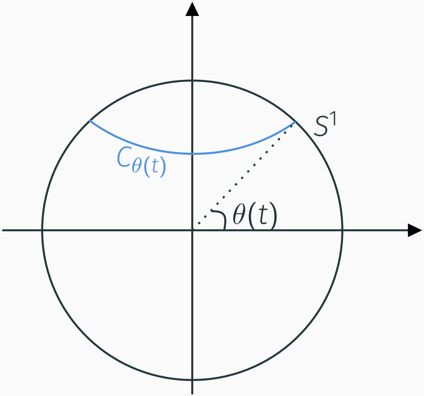

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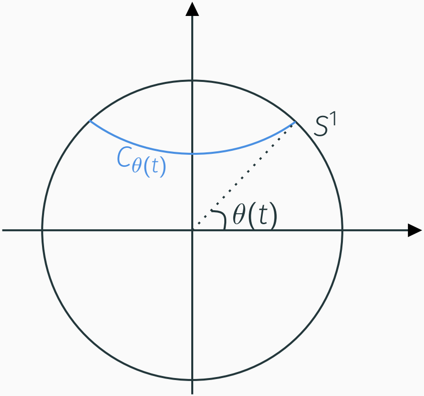

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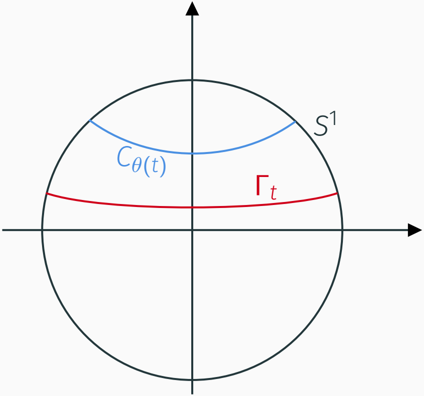

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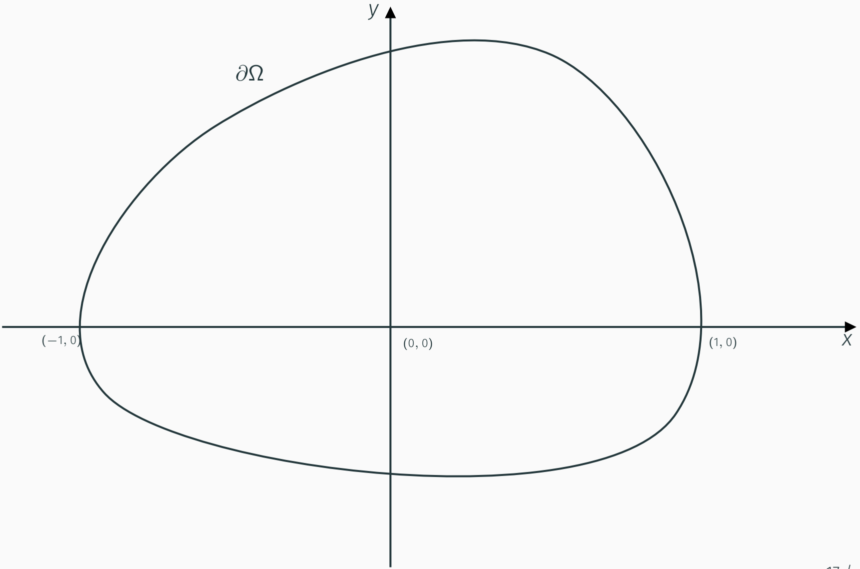

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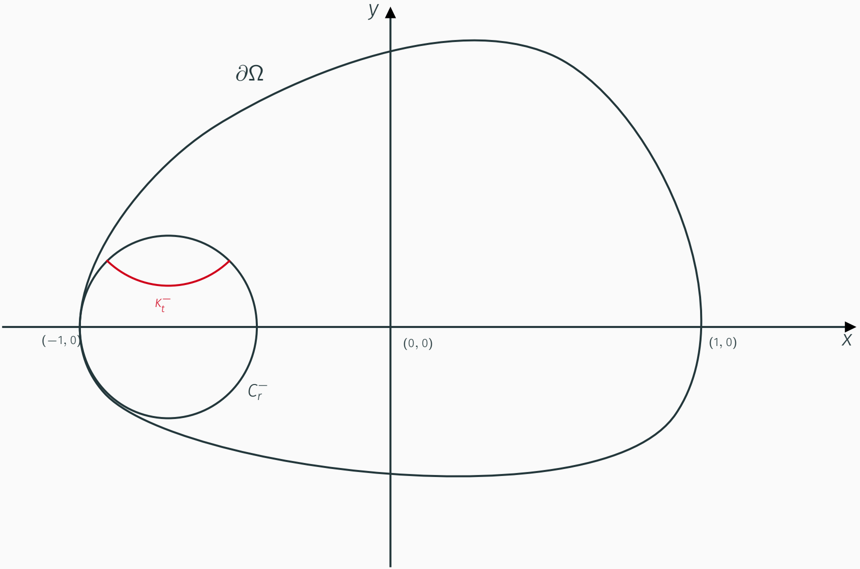

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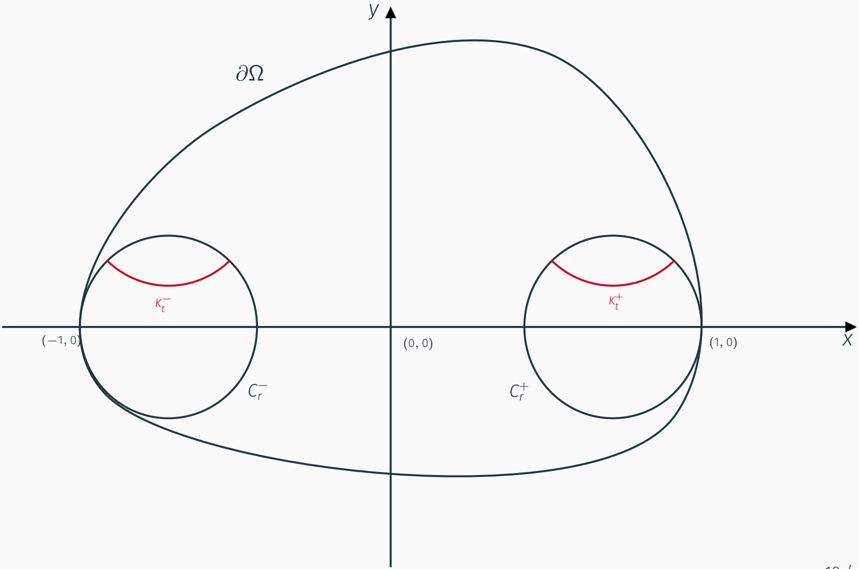

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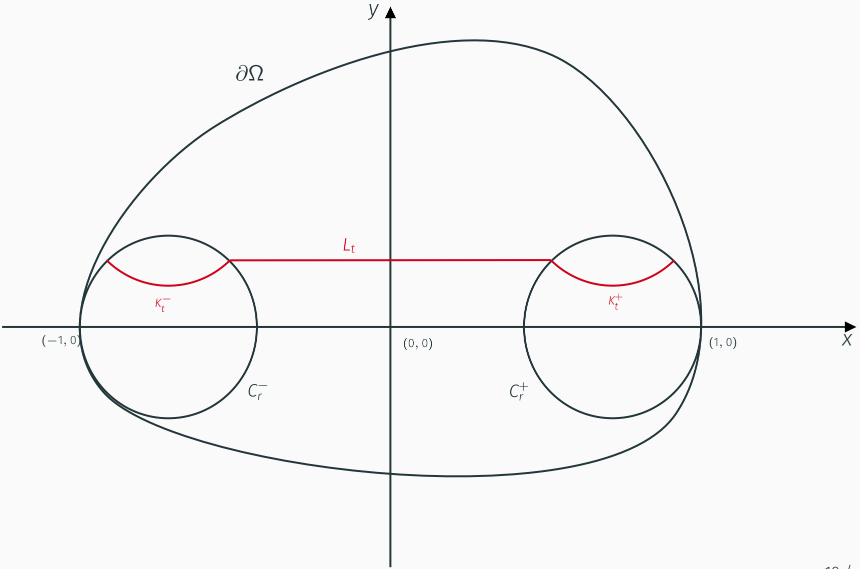

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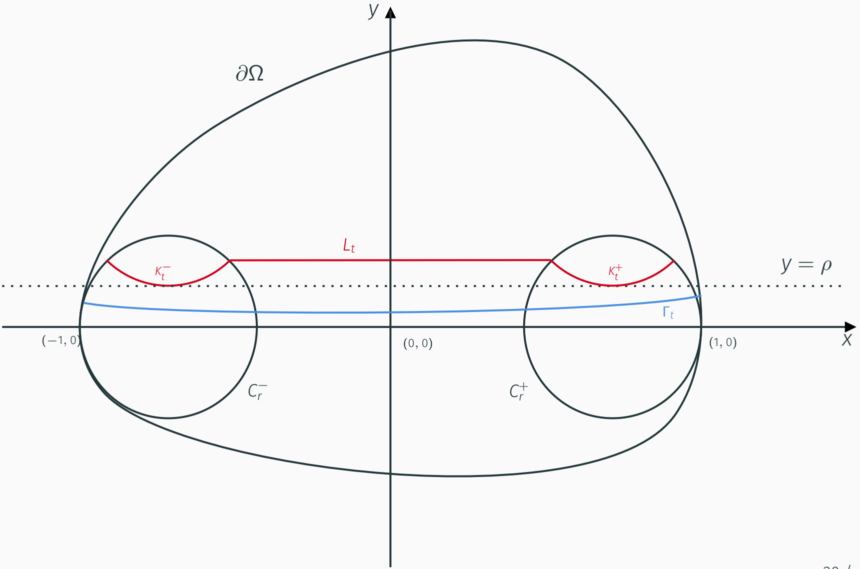

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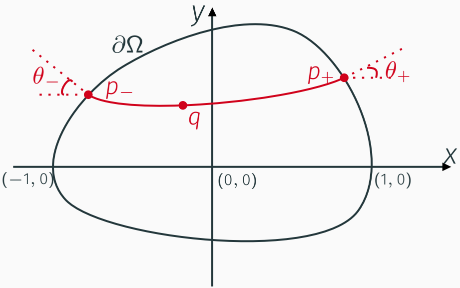
- In order to take the limit of these flows, we need uniform estimates for the curvature and their derivatives.
- It suffices to prove gradient estimates for the height function.

Gradient Estimates

Parameterise Γ_t^ρ by turning angle $\gamma : [-\theta_-, \theta_+] \to \overline{\Omega}$.



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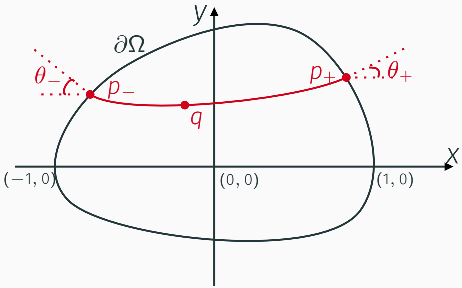

$$2 \geq \langle p_+ - q, e_1 \rangle = \int_{\theta(q)}^{\theta_+} \frac{\cos(u)}{\kappa(u)} du \geq \frac{\sin \theta_+ - \sin(\theta(q))}{\kappa(p_+)}$$

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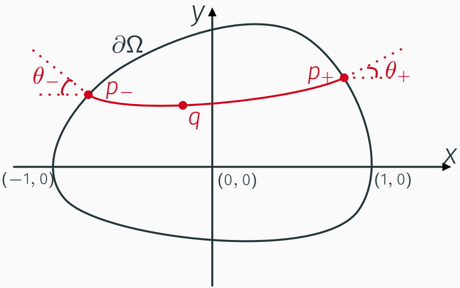

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$$2 \geq \langle q - p_-, e_1 \rangle = \int_{-\theta_-}^{\theta(q)} \frac{\cos(u)}{\kappa(u)} du \geq \frac{\sin(\theta(q)) + \sin\theta_-}{\kappa(p_-)}$$

Gradient Estimates

These inequalities, along the evolution equations for turning angle, gives an ODE inequality for all $t < t_0$:

$$\frac{d(\theta_+ + \theta_-)}{dt} \geq 2r\tan(\frac{\theta_+ + \theta_-}{2})$$

which, after integrating yields

$$\sin\left(\frac{\theta_+ + \theta_-}{2}\right) \leq Ce^{rt}$$

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Barriers + gradient bounds imply ([Stahl]) that as $\rho \to 0$, $\{\Gamma_t^\rho\}_{t\in[-\alpha_\rho,0)} \to \{\Gamma_t\}_{t\in(-\infty,0)}$.

Uniqueness of the Ancient Solution

Height Asymptotics on the Constructed Solution

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- We choose our initial curves Γ^ρ to be sections of time-slices of Angenent ovals

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- Can show that as $\rho\to 0$, $\lambda\to\lambda_0$, where λ_0 is the largest solution to

$$\lambda_0^2 - \lambda_0(\kappa_1+\kappa_2)\coth 2\lambda_0 + \kappa_1\kappa_2 = 0,$$

  and κ_1, κ_2 are the curvatures of $\partial\Omega$ at e_1 and $-e_1$ respectively.

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After some *involved* maximum principle arguments, we can show that on the constructed solution

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Since with a graph parametrisation, $\log(y)_t = \frac{\kappa}{y\cos\theta}$, one can show that the limit

$$\lim_{t\to-\infty} e^{-\lambda_0^2 t} y(x,t)$$

exists *on the constructed ancient solution* and is strictly positive for each $x \in (-1,1)$.

Height Asymptotics on Any Solution

Lemma

Any two ancient solutions to the free boundary curve shortening flow in $\Omega \cap \{y \geq 0\}$ *must intersect for all sufficiently negative time.*

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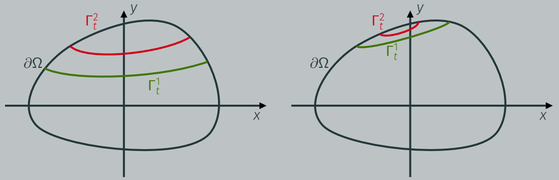

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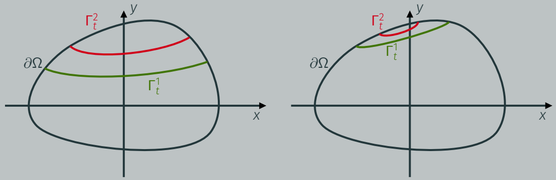

This violates the avoidance principle.

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Using this comparison, we can conclude that

$$\lim_{t \to -\infty} e^{-\lambda_0^2 t} y(x, t)$$

exists on *any* ancient solution.

Eigenvalue Problem

Now consider the rescaled height function:

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Now consider the rescaled height function:

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Since this is uniformly bounded, there exists a (weak*) subsequential limit y^{∞} which solves the problem:

$$\begin{cases} y_t^{\infty} = y_{xx}^{\infty} \ \text{ in } \ [-1,1] \\ y_x^{\infty}(\pm 1) = \pm y^{\infty} \kappa(\pm e_1) \end{cases}$$

Eigenvalue Problem

Separation of variables leads us to consider the problem:

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Analysing eigenvalues $\Rightarrow$

$$y^\infty(x,t) = Ae^{\lambda_0^2 t}\left(\cosh\lambda_0 x + \frac{\kappa_1 - \kappa_2}{2\lambda_0 - (\kappa_1+\kappa_2)\tanh\lambda_0}\sinh\lambda_0 x\right)$$

where κ_1, κ_2 are the curvatures of $\partial\Omega$ at e_1 and $-e_1$ respectively.

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Every term on the RHS is independent of the ancient solution!

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Taking the rescaled limit as $t \to -\infty$:

$$e^{-\lambda_0^2 t} y^\tau(x,t) \to A e^{\lambda_0^2 \tau} \left(\cosh(\lambda_0 x) + \frac{\kappa_1 - \kappa_2}{2\lambda_0 - (\kappa_1 + \kappa_2)\tanh \lambda_0} \sinh(\lambda_0 x) \right)$$

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Therefore $\{\Gamma^\tau_t\}$ **lies above** $\{\Gamma_t\}$ for all $\tau > 0$.

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Therefore the solution is unique!!

Uniqueness

Theorem (B.-Bourni-Catron '24)

Let Ω be a bounded, connected, strictly convex domain in $\mathbb{R}^2$. Modulo time-translation, for each diameter of Ω (that is, a line segment intersecting the boundary of Ω orthogonally), there exists precisely two convex, locally uniformly convex, ancient solutions to the free boundary curve shortening flow in Ω, one lying on each side of the diameter.

Thank You.