The Free Boundary Curve Shortening Flow and its Ancient Solutions

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AMS Special Session on Geometric and Non-Smooth Analysis

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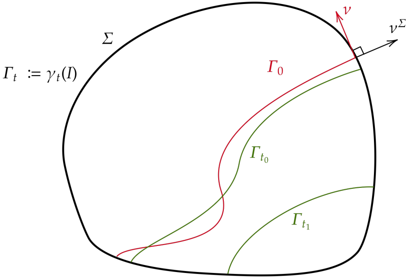

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- (Bourni-B.-Langford '26) Whenever $\mathcal{M}_t$ converges to a point with rescaled image the unit semi-circle, it does so in the C^k-topology (for any $k \geq 0$) at a rate no slower than $(T - t)^{1-\delta}$ for any $\delta > 0$.

Classification Results for the Curve Shortening Flow

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- (Bourni-Langford-Tinaglia ’19) The only convex-embedded ancient solutions to the curve shortening flow are the stationary lines, shrinking circles, grim reapers and Angenant ovals.

Ancient Solutions to FBCSF in Compact Domains

Ancient Solutions in Compact Free-Boundaries

Theorem (Bourni-Langford)

If $\Sigma = S^1$, then up to rotation and time translation, there exists a unique convex ancient solution to the free boundary curve shortening flow.

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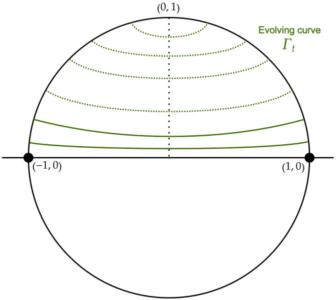

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Theorem (Bourni-B.-Catron '25)

For any compact free boundary Σ, and any diameter D, we have exactly two convex ancient solutions to the free boundary curve shortening flow which converge to D as $t \to -\infty$.

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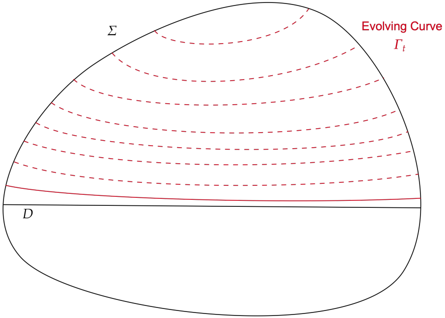

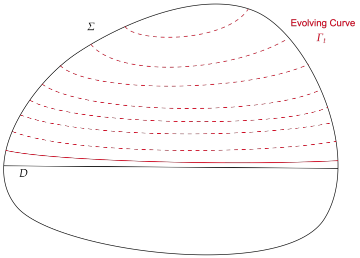

Proof Idea:

- Existence: Take a sequence of old-but-not-ancient solutions $\{\Gamma_t\}_{t\in[\alpha_\rho,0)}$.

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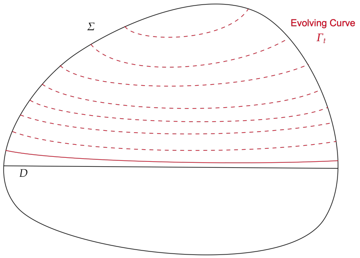

Proof Idea:

- Existence: Take a sequence of old-but-not-ancient solutions $\{\Gamma_t\}_{t\in[\alpha_\rho,0)}$.
- Uniqueness: Look at the asymptotic behavior of the quantity $e^{-\lambda_0^2 t}y(x,t)$, and use the strong maximum principle.

Non-Compact Ancient Solutions in Non-Compact Free Boundaries

The Non-Compact Setting

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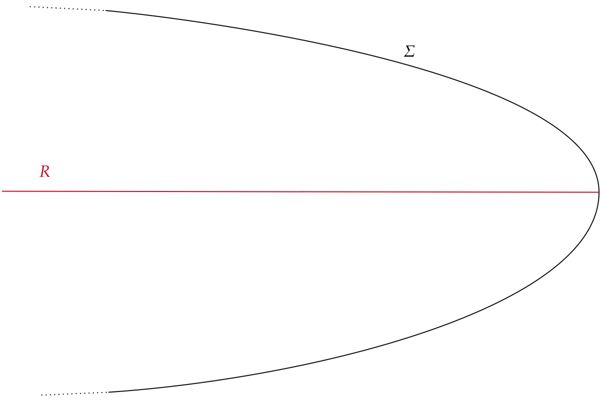

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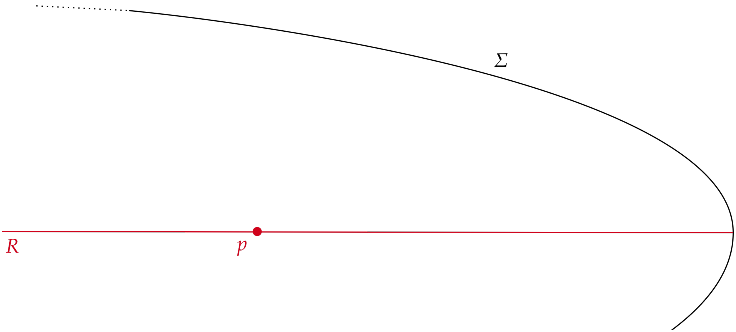

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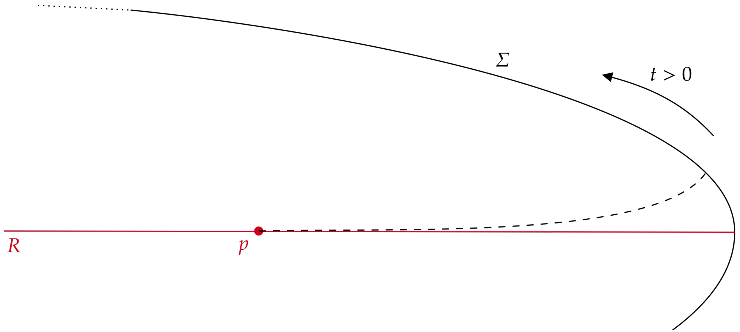

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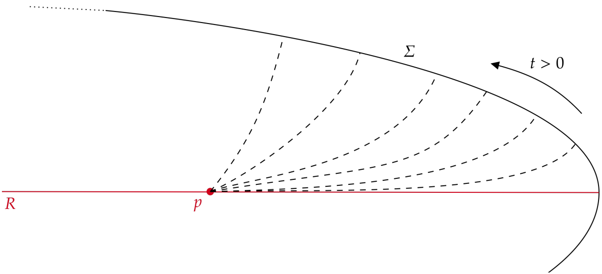

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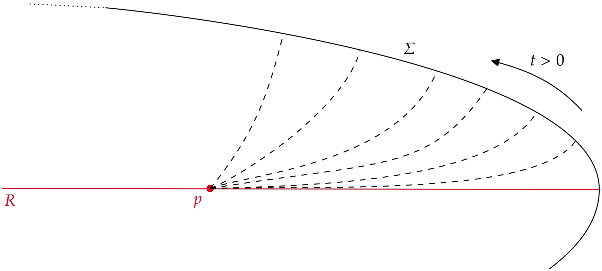

For all p *“sufficiently large”*, by adapting the same methods as the compact case, we can find a unique ancient solution to this Dirichlet-Neumann CSF.

The Non-Compact Result

Theorem (Bourni-B.-Langford)

Given any such Σ and R, there are precisely two convex, non-compact ancient solutions to the free boundary curve shortening flow which converges to R as $t \to -\infty$.

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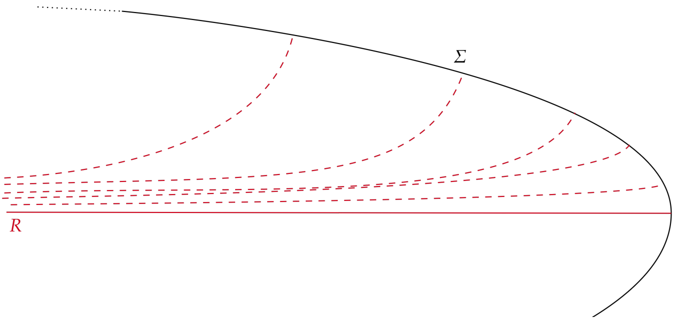

Compact Ancient Solutions in Cones

Situation 1: *Small Cones*

Consider a cone with vertex at the origin, making an angle $\omega \leq \frac{\pi}{2}$:

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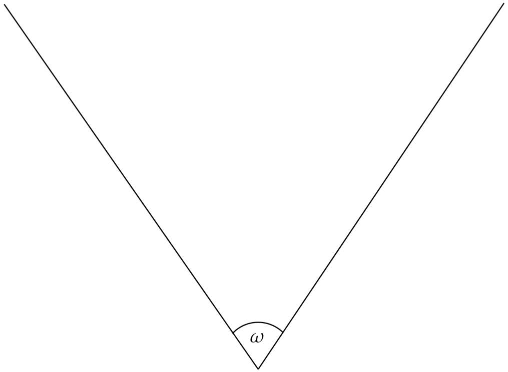

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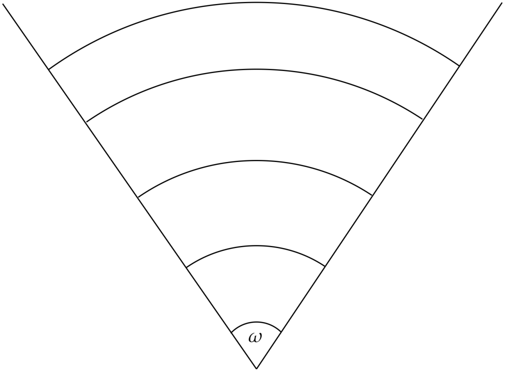

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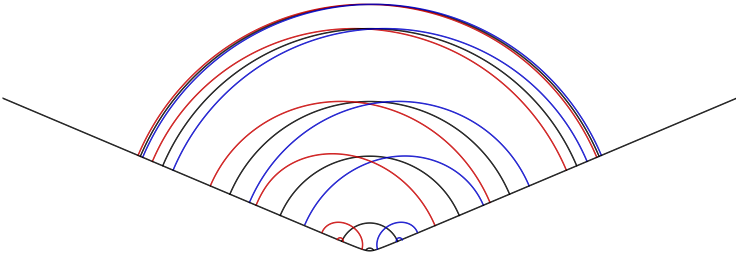
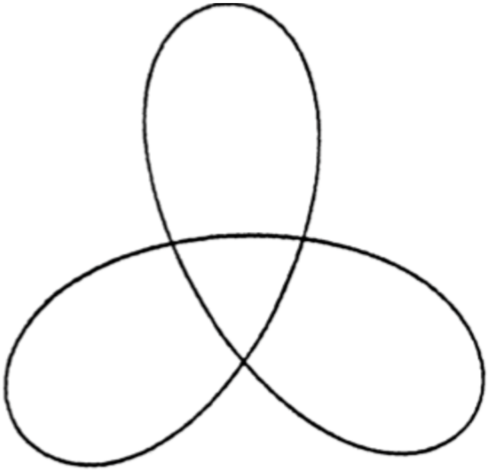


Situation 3: Intermediate Cones



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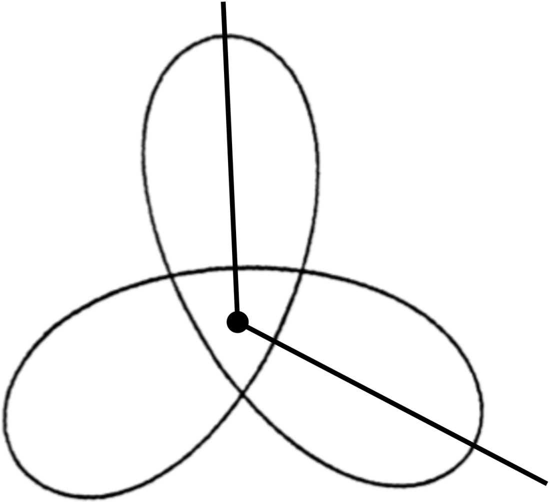

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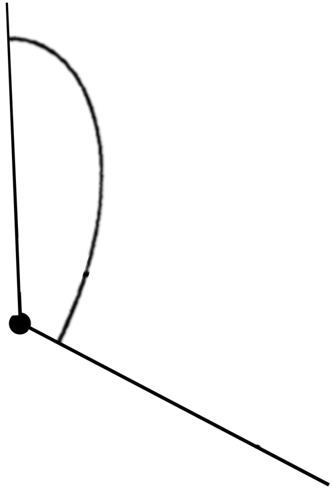

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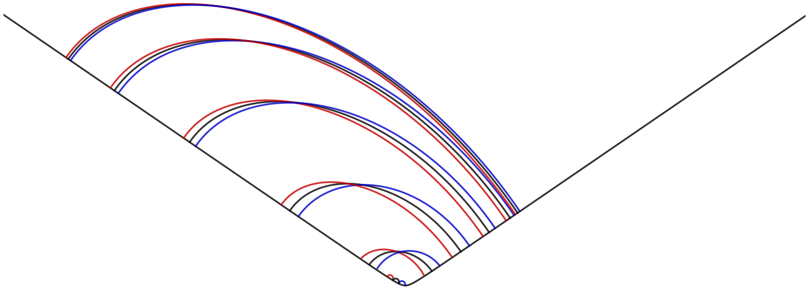

Cones: An Intuition

The *normalised* support function σ satisfies the following problem

$$\begin{cases} \partial_t \sigma = \sigma - (\sigma_{\theta\theta} - \sigma)^{-1} & \theta \in [0, \omega] \\ \sigma_\theta = 0 & \theta = 0, \omega. \end{cases}$$

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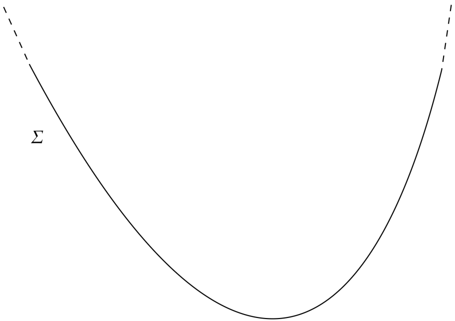
The eigenstates are given by

$$v_k(\theta, t) = \mathrm{e}^{-\mu_k t} \cos\left(\tfrac{k\pi}{\omega}\theta\right), \quad \mu_k = \tfrac{k^2\pi^2}{\omega^2} - 2\,, \quad k = 0, 1, 2, \ldots\,.$$

Compact Ancient Solutions in Non-Compact Domains

The Compact, Non-Compact Result

Consider a non-compact, convex curve Σ bounding a convex region Ω.



Let $\pi - \omega$ be the total turning angle of Σ.

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This leads to the following result:

1. If $\omega \in (0, \frac{\pi}{2}]$, then there is a unique compact, convex, ancient solution in Ω.
2. If $\omega \in (\frac{\pi}{2}, \frac{\pi}{\sqrt{2}})$, then for each $p \in \Sigma$ there are at most 3 compact, convex ancient solutions inside Ω which get extinct at p. These are characterized by their backwards limit.
3. If $\Omega \in [\frac{\pi}{\sqrt{2}}, \pi)$, then for each $p \in \Sigma$, there exists a unique compact, convex ancient solution which gets extinct at p.

Thank You!