Stability of Ancient Rays

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AMS Special Session on Riemannian Geometry and Symmetry II

Free Boundary Curve Shortening Flow

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Recall free boundary curve shortening flow:

$\gamma_t : I \to \mathbb{R}^2$ is a one-parameter family of curves such that

$$\begin{cases} \frac{\partial \gamma}{\partial t} = -\kappa \nu \\ \langle \nu, \nu^{\Sigma} \circ \gamma \rangle = 0. \end{cases}$$

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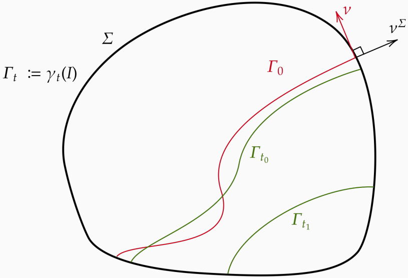

Ancient Solutions in Compact Free-Boundaries

Theorem (Bourni-Langford)

If $\Sigma = S^1$, then up to rotation and time translation, there exists a unique convex ancient solution to the free boundary curve shortening flow.

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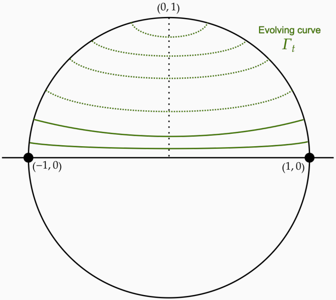

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For any compact free boundary Σ, and any diameter D, we have exactly two convex ancient solutions to the free boundary curve shortening flow which converge to D as $t \to -\infty$.

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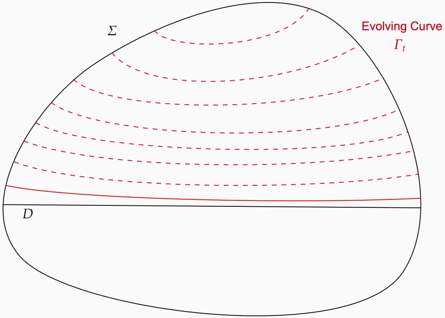
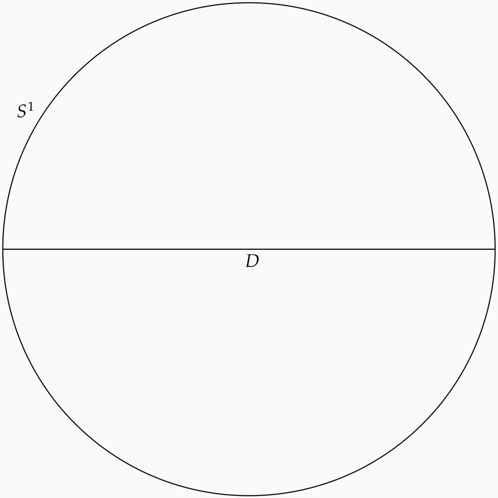


The Stability of the Diameters

Model Case: The Circle



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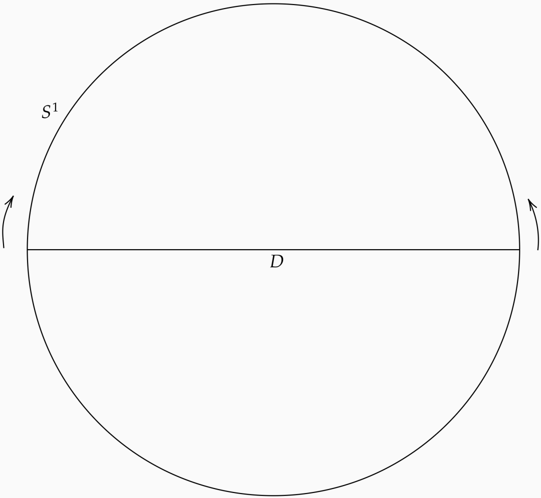

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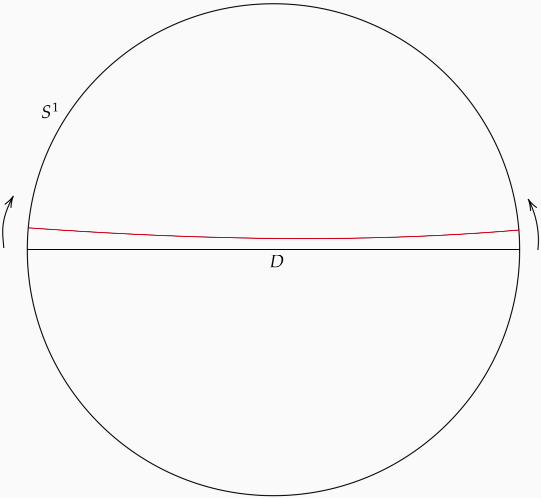

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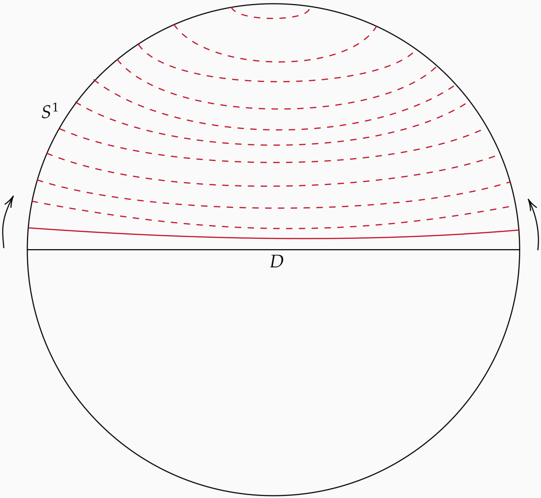

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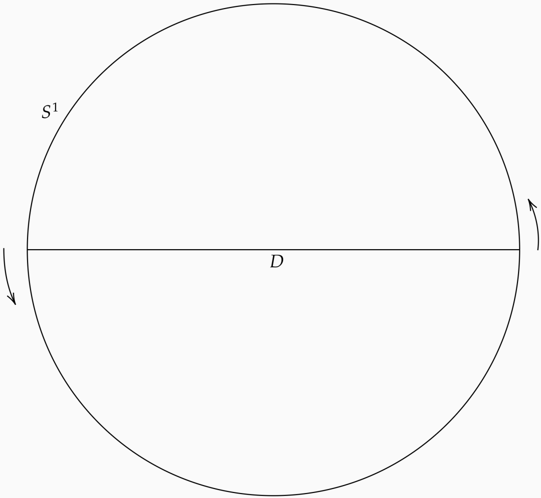

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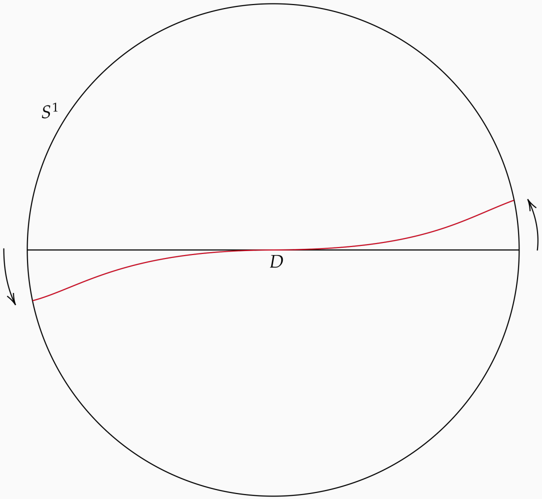

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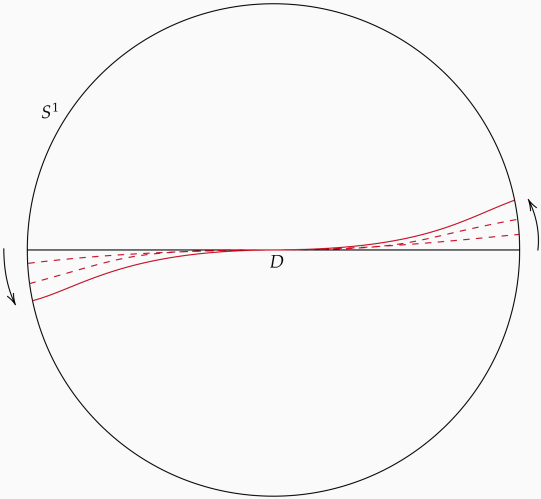

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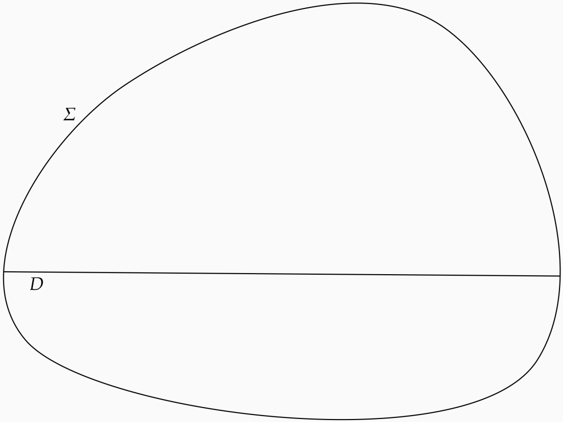

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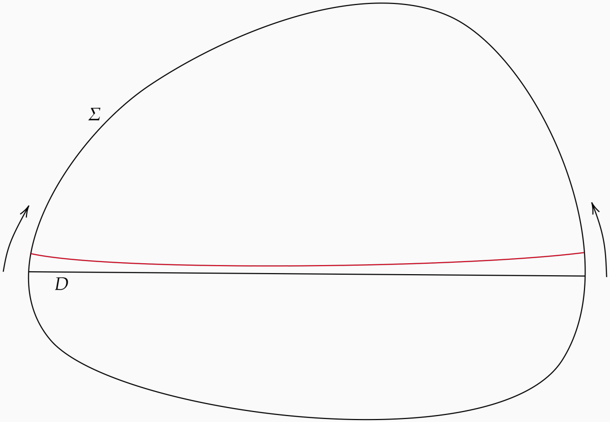

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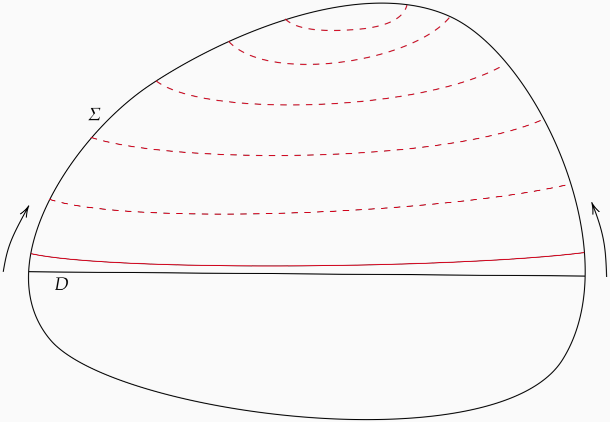

How do we quantify the so-called *“stability”*?

The Parametric Height Function

Assume that D lies on the x-axis, and let $y := \langle \gamma_t, e_2 \rangle$.

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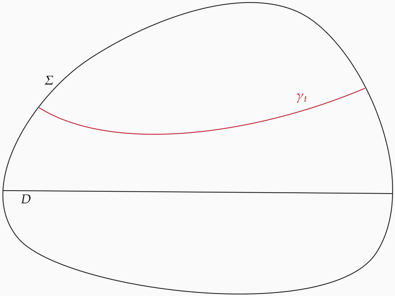

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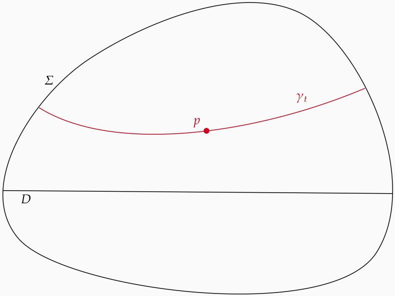

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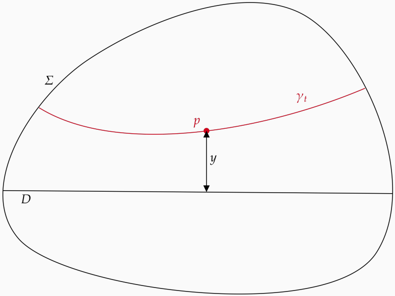

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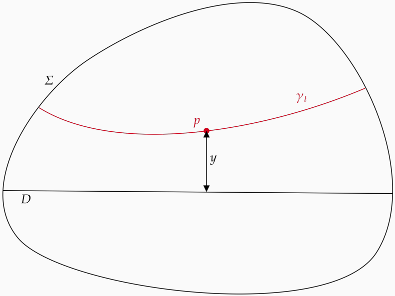

The function y satisfies the following equation:

$$\begin{cases} y_t = y_{ss} & \text{int}(\Gamma_t), \\ \langle \nabla y, e_2 \rangle = \left\langle \nu^{\Sigma}, e_2 \right\rangle & \partial \Gamma_t. \end{cases}$$

If we linearise around the diameter D, this becomes

$$\begin{cases} u_t = u_{xx} & x \in D \\ u_x(\pm e_1) = \pm\kappa^{\Sigma}(\pm e_1)u(\pm e_1) & x \in \partial D. \end{cases}$$

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How does this relate to ancient solutions to the free boundary curve shortening flow?

The Relationship to the Ancient Solutions

Let $\{\gamma_t\}_{t\in(-\infty,0)}$ be a convex ancient solution to the free boundary curve shortening flow converging to the diameter D.

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On *any* convex ancient solution, one can show that

$$e^{-\lambda_0^2 t} y(x,t) \overset{t\to-\infty}{\longrightarrow} u(x,t)$$

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$$u(x,t) = A\left(\cosh \lambda_0 x + \frac{\kappa^{\Sigma}(e_1) - \kappa^{\Sigma}(-e_1)}{2\lambda_0 - (\kappa^{\Sigma}(e_1) + \kappa^{\Sigma}(-e_1)) \tanh \lambda_0} \sinh \lambda_0 x\right)$$

Non-Compact Ancient Solutions

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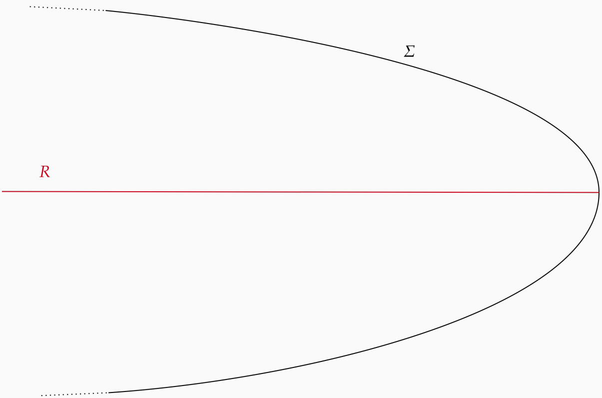

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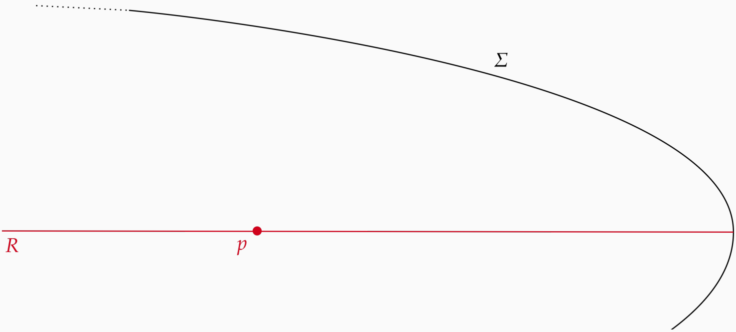

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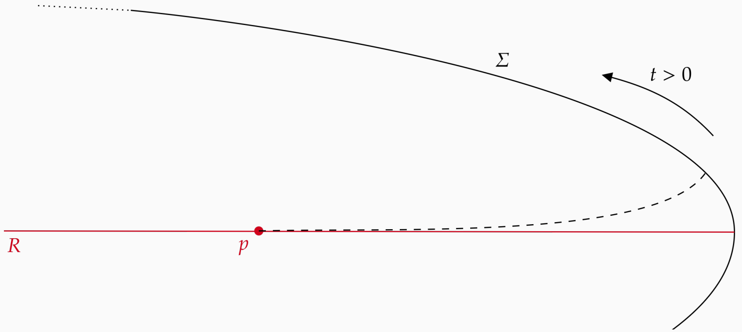

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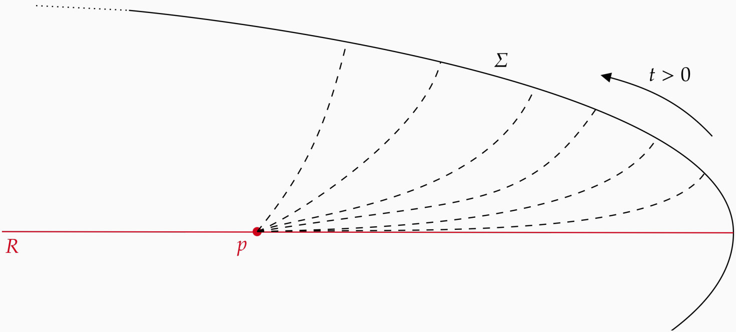

Linearised Height for Mixed Boundary Flows

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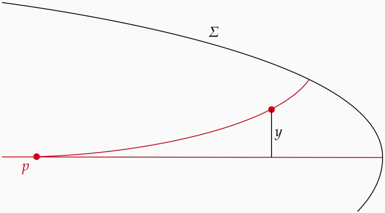

$$\begin{cases} y_t = y_{ss} & x \in R \\ \langle \nabla y, e_2 \rangle = \left\langle \nu^{\Sigma}, e_2 \right\rangle & x \in R \cap \Sigma \\ y(p) = 0 & x = p. \end{cases}$$

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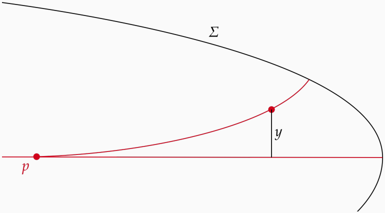

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Linearising about the portion of R between p and Σ:

$$\begin{cases} \partial_t u_p = \partial^2_{xx} u_p & x \in R \\ \partial_x u_p(0) = \kappa^{\Sigma}(0) u(0) & 0 \in R \cap \Sigma \\ u_p(p) = 0 & x = p. \end{cases}$$

Mixed Boundary Stability

For a given $p \in R$, let $\{\gamma_t^p\}_{t \in (-\infty, 0)}$ be an ancient solution to the mixed boundary curve shortening flow converging to the portion of R between p and Σ.

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As before, we can find the following limit on any ancient solution to the mixed boundary problem

$$\lim_{t \to -\infty} e^{-\lambda(p)t} y(x,t) \to u_p(x,t).$$

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In this case, $u_p(x,t)$ is a solution to the corresponding mixed boundary linearised problem:

$$\begin{cases} \partial_t u_p = \partial_{xx}^2 u_p & x \in R \\ \partial_x u_p(0) = \kappa^\Sigma(0)u(0) & 0 \in R\cap\Sigma \\ u_p(p) = 0 & x = p. \end{cases}$$

Mixed Boundary Stability

Letting $|p| \to \infty$, let $\{\gamma_t^\infty\}_{t \in (-\infty, 0)}$ be an ancient solution to the free boundary curve shortening flow converging to R.

As before, we can find the following limit on any ancient solution to the mixed boundary problem

$$\lim_{t \to -\infty} e^{-\lambda(\infty)t} y(x,t) \to u_\infty(x,t).$$

In this case, $u_\infty(x,t)$ is a solution to the corresponding mixed boundary linearised problem:

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The Non-Compact Result

Theorem (Bourni-B.-Langford)

Given any such Σ and R, there are precisely two convex ancient solutions to the free boundary curve shortening flow which converges to R as $t \to -\infty$.

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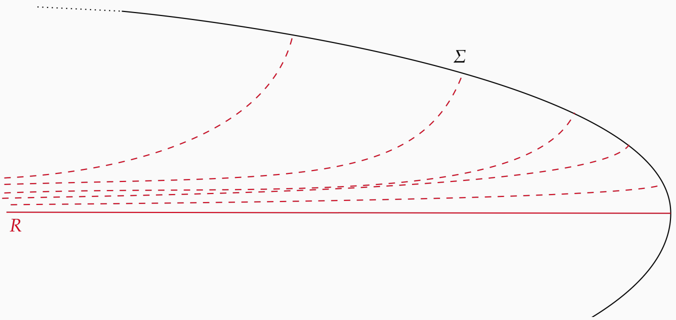

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Thank You!